Canonical Formulation of pp-waves

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Abstract

We construct a Hamiltonian formulation for the class of plane-fronted gravitational waves with parallel rays (pp-waves). Because of the existence of a light-like Killing vector, the dynamics is effectively reduced to a 2+1 evolution with "time" chosen to be light-like. In spite of the vanishing action this allows us to geometrically identify a symplectic form as well as dynamical Hamiltonian, thus casting the system into canonical form.

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Introduction

Plane-fronted gravitational waves with parallel rays (pp-waves)are considered as highly idealized wave phenomena which have been extensively studied in General Relativity and related areas. These geometries are characterized by the existence of a covariantly constant null vector field p^a. In a by now classical work Jordan, Ehlers and Kundt [1] have given a complete classification of the pp-vacuum solutions to the Einstein field equations in terms of their spacetime symmetries.

The special class of the so called impulsive pp-waves (which were excluded in [1] but treated in [2, 3]), i.e. geometries which are flat space everywhere except of a single null-hypersurface generated by p^a have been found to describe both the ultra-relativistic (null-limit) of (stationary) black holes as well as the gravitational field generated by massless particles [4]. This has led to a semiclassical investigation of particle scattering at ultrahigh (Planckian) energies within these backgrounds which displays amplitudes similar to those appearing in String theory [5, 6]. Also, pp-waves belong to the class of algebraic special solutions of Petrov type N. Moreover, all invariants formed from the curvature tensor vanish identically. This property has made them a candidate as an exact background for the consistent propagation of strings [7]. Due to the above mentioned richness it seems worthwhile to investigate a possible quantization of this family of geometries. From the canonical point of view due to the frozen degrees of freedom such a quantization should yield a midi-superspace model [8]. Unfortunately the vanishing of the action for the whole class of pp-waves does not allow a "straightforward" Hamiltonian formulation, which relies on Legendre transformation of the Lagrangian. However, upon a careful analysis of the equations of motion, we succeed in the construction of a symplectic form as well as a Hamiltonian (vector field) which generates the evolution. The investigation of this structure will be the aim of the present work.

Our work is organized as follows: After stating our conventions, we briefly review the timelike situation in terms of a Gaussian decomposition of an arbitrary metric. This section is mainly devoted to familiarize the reader with the concepts used in the lightlike situation. Section two derives a similar evolution formulation for the lightlike (pp-wave) setting, which yields an effectively 2+1 dimensional situation. In section three we discuss the propagation of the constraints of the pp-wave system. Finally, section four casts the dynamical system into Hamiltonian form, as a pre-requisite for quantization.

0 Conventions

Our conventions with respect to metric and covariant derivatives follow closely that of Wald [9] (with $a, b\ldots$ referring to abstract indices)

$$\begin{aligned}
\nabla_a g_{bc} = 0 \text{ and } [\nabla_a, \nabla_b]f = 0 \\
[\nabla_a, \nabla_b]v^b = R^b{}_{mab}v^m \\
R_{ab} = R^c{}_{acb}, \;\; R = g^{ab}R_{ab}
\end{aligned} \tag{1}$$

The signature of the (spacetime) metric is taken to be $(- + ++)$. In terms of (normalized) tetrads we have

$$g_{ab} = \eta_{\alpha\beta}e^\alpha_a e^\beta_b \qquad g^{ab} = \eta^{\alpha\beta}E^a_\alpha E^b_\beta \tag{2}$$

where E^a_α and e^α_a denote dual frames. The Cartan structure relations for the spin-connection $\omega^\alpha{}_{\beta a}$, the Riemann two-form $R^\alpha{}_{\beta ab}$ and the Ricci one-form $R^\alpha{}_a$ become

$$\begin{aligned}
de^\alpha = -\omega^\alpha{}_\beta e^\beta \\
R^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \omega^\gamma{}_\beta \\
R_\alpha = E_\beta \lrcorner R^\beta{}_\alpha \\
R = E_\beta \lrcorner R^\beta
\end{aligned} \tag{3}$$

where the skew (wedge) product in the above relations is implicitly understood. The hook $\lrcorner$ denotes the contraction of a p-form with a vector field

1 Einstein equations in Gaussian coordinates

In order to gain some familiarity with the approach used for pp-waves let us begin with the well-known $3+1$decomposition of the Einstein equations in terms of Gaussian coordinates

$$ds^2 = -dt^2 + h_{ij}(x,t)dx^i dx^j \tag{4}$$

where the $t = const$ surfaces denote the spacelike slices of the Gaussian coordinate system. Using a canonically adapted tetrad

$$e^\alpha = (dt, \tilde{e}^i(x,t)) \qquad\qquad E_\alpha = (\partial_t, \tilde{E}_i(t,x)) \tag{5}$$

the corresponding connection is derived from the structure equations (in the expression above we have explicitly exhibited the parametric t- dependence, whose derivatives will be denoted by a dot in the following, e.g. $\partial_t \tilde{e}^i = \dot{\tilde{e}}^i$)

$$
\begin{aligned}
&d\tilde{e}^i = -\tilde{\omega}^i{}_j \tilde{e}^j + dt\dot{\tilde{e}}^i = -(\tilde{\omega}^i{}_j + F^i{}_j dt)\tilde{e}^j - K^i{}_j \tilde{e}^j dt \\
&\omega^i{}_j = \tilde{\omega}^i{}_j + F^i{}_j dt, \quad \omega^i{}_t = K^i{}_j \tilde{e}^j,
\end{aligned} \tag{6}
$$

where we have decomposed $\dot{\tilde{e}}^t$ with respect to $\tilde{e}^i$ and split the corresponding coefficient matrix $\tilde{E}_i \lrcorner \dot{\tilde{e}}^i$ into its symmetric and antisymmetric part respectively

$$
K^i{}_j = \frac{1}{2}(\tilde{E}_j \lrcorner \dot{\tilde{e}}^i + \tilde{E}^i \lrcorner \dot{\tilde{e}}_j) \quad F^i{}_j = \frac{1}{2}(\tilde{E}_j \lrcorner \dot{\tilde{e}}^i - \tilde{E}^i \lrcorner \dot{\tilde{e}}_j).
$$

From this we derive the components of the Riemann 2-form

$$
\begin{aligned}
R^i{}_j &= d\omega^i{}_j + \omega^i{}_l \omega^l{}_j + \omega^i{}_t \omega^t{}_j \\
&= \tilde{R}^i{}_j + dt\dot{\tilde{\omega}}^i{}_j + \tilde{D}F^i{}_j dt + K^i{}_l K_{jm} \tilde{e}^l \tilde{e}^m, \\
R^i{}_t &= d\omega^i{}_t + \omega^i{}_j \omega^j{}_t \\
&= \tilde{D}K^i{}_j \tilde{e}^j + (\dot{K}^i{}_j + K^i{}_l K^l{}_j - K^i{}_l F^l{}_j + F^i{}_l K^l{}_j) dt \tilde{e}^j,
\end{aligned} \tag{7}
$$

and finally the Ricci 1-forms

$$
\begin{aligned}
R_t &= E_i \lrcorner R^i{}_t \\
&= (\tilde{D}_i K^i{}_j - \tilde{D}_j K)\tilde{e}^j - (\dot{K} + K^i{}_j K^j{}_i) dt, \\
R_i &= E_t \lrcorner R^t{}_i + E_j \lrcorner R^j{}_i \\
&= (K_{ij} + KK_{ij} - K_{il}F^l{}_j + F_{il}K^l{}_j)\tilde{e}^j + \tilde{R}_i - dt\tilde{E}_j \lrcorner \dot{\tilde{\omega}}^j{}_i + \tilde{D}_j F^j{}_i dt.
\end{aligned} \tag{8}
$$

where

$$
\tilde{D}v^i = \tilde{d}v^i + \tilde{\omega}^i{}_j \tilde{v}^j
$$

denotes the exterior covariant derivative relative to the $t = const.$ surfaces. The Einstein equations reduce to

$$
\begin{aligned}
\tilde{D}_i K^i{}_j - \tilde{D}_j K &= 0, \\
\dot{K} + K^i{}_j K^j{}_i &= 0, \\
\dot{K}_{ij} + KK_{ij} - K_{il}F^l{}_j + F_{il}K^l{}_j + \tilde{R}_{ij} &= 0,
\end{aligned} \tag{9}
$$

where we took into account that

$$
\tilde{D}_i K^i{}_j - \tilde{D}_j K - \tilde{D}_i F^i{}_j = \tilde{E}_i \lrcorner \dot{\tilde{\omega}}^i{}_j
$$

which follows from taking the "time" derivative of three-dimensional structure equation $\tilde{d}\tilde{e}^i = -\tilde{\omega}^i{}_j\tilde{e}^j$. The first equation and the difference between the second and the trace of the third equation are just the well-known momentum and Hamilton constraints of General Relativity respectively

$$\begin{aligned} \tilde{D}_i K^i{}_j - \tilde{D}_j K &= 0, \\ K^i{}_j K^j{}_i - K^2 - \tilde{R} &= 0. \end{aligned} \tag{10}$$

which are constraints on the initial data whereas dynamics is contained in

$$\dot{K}_{ij} + KK_{ij} - K_{il}F^l{}_j + F_{il}K^l{}_j + \tilde{R}_{ij} = 0. \tag{11}$$

The arbitrariness of the choice of triad $\tilde{e}^i_a$ reflects itself in the appearance of the antisymmetric "field-strength" $F^i{}_j$. However, if we re-express the equations in terms of the 3-metric $h_{ab} = \delta_{ij}e^i{}_a e^j{}_b$ we find

$$\begin{aligned} \dot{h}_{ab} &= \delta_{ij}(\dot{\tilde{e}}^i{}_a\tilde{e}^j{}_b + \tilde{e}^i{}_a\dot{\tilde{e}}^j{}_b) \\ &= \delta_{ij}((K^i{}_l - F^i{}_l)\tilde{e}^l{}_a\tilde{e}^j{}_b + \tilde{e}^i{}_a(K^j{}_l - F^j{}_l)\tilde{e}^l{}_b) \\ &= 2K_{ij}\tilde{e}^i{}_a\tilde{e}^j{}_b = 2K_{ab}. \end{aligned} \tag{12}$$

Together with the other equation we therefore have a decomposition of the Einstein equations into dynamical and constraint equations.

$$\begin{aligned} \dot{h}_{ab} = 2K_{ab} \qquad & D_a K^a{}_b - D_b K = 0 \\ \dot{K}_{ab} = -KK_{ab} - \tilde{R}_{ab} \qquad & K^a{}_b K^b{}_a - K^2 - \tilde{R} = 0 \end{aligned} \tag{13}$$

(where D_a denotes the Levi-Civita derivative of h_{ab})

2 2+1(+1) decomposition of pp-waves

In this section we describe an analogous (quasi-Gaussian) decompositon for pp-wave geometries, i.e. metrics characterized by the existence of a covariantly constant null vector-field

$$ds^2 = -2dudv + \sigma_{ij}(x,u)dx^i dx^j \tag{14}$$

Since $p^a = \partial^a_v$ generates a Killing symmetry we are dealing effectively with a 2+1 decomposition of a system dimensionally reduced. However, the "time" direction is chosen to be lightlike. This null direction is geometrically singled

out by being orthogonal to the (arbitrarily) chosen (spacelike) 2-slices. The canonically adapted tetrad is given by

$$e^\alpha = (du, dv, \tilde{e}^i(u,x)) \qquad\qquad E_\alpha = (\partial_u, \partial_v, \tilde{E}_i(x,u)) \tag{15}$$

(As in the timelike-case, a dot will denote the derivative with respect to the "time"-parameter u)

$$\begin{array}{l} d\tilde{e}^i = -\tilde{\omega}^i{}_j\tilde{e}^j + du\dot{\tilde{e}}^i = -(\tilde{\omega}^i{}_j + F^i{}_j du)\tilde{e}^j - K^i{}_j\tilde{e}^j du \\ \omega^i{}_j = \tilde{\omega}^i{}_j + F^i{}_j du \quad \omega^i{}_u = K^i{}_j\tilde{e}^j \end{array} \tag{16}$$

where we used the decomposition of $\tilde{E}_i\lrcorner\dot{\tilde{e}}^j := \tilde{E}^a_i\dot{\tilde{e}}^j_a$ into symmetric and antisymmetric parts, respectively

$$K^i{}_j = \frac{1}{2}(\tilde{E}_j\lrcorner\dot{\tilde{e}}^i + \tilde{E}^i\lrcorner\dot{\tilde{e}}_j) \qquad F^i{}_j = \frac{1}{2}(\tilde{E}_j\lrcorner\dot{\tilde{e}}^i - \tilde{E}^i\lrcorner\dot{\tilde{e}}_j).$$

Therefore the non-vanishing components of the curvature 2-form are

$$\begin{array}{rcl} R^i{}_j &=& d\omega^i{}_j + \omega^i{}_l\omega^l{}_j \\ &=& \tilde{R}^i{}_j + du\dot{\tilde{\omega}}^i{}_j + \tilde{D}F^i{}_j du \\ R^i{}_u &=& d\omega^i{}_u + \omega^i{}_j\omega^j{}_u \\ &=& \tilde{D}K^i{}_j\tilde{e}^j + (\dot{K}^i{}_j + K^i{}_lK^l{}_j - K^i{}_lF^l{}_j + F^i{}_lK^l{}_j)du\tilde{e}^j \\ where && \tilde{D}\tilde{v}^i = \tilde{d}\tilde{v}^i + \tilde{\omega}^i{}_j\tilde{v}^j \end{array} \tag{17}$$

Together with identity

$$(\tilde{D}K^i{}_j - \tilde{D}F^i{}_j)\tilde{e}^j + \dot{\tilde{\omega}}^i{}_j\tilde{e}^j = 0$$

one easily obtains the Ricci one-form

$$\begin{array}{rcl} R_u &=& E_i\lrcorner R^i{}_u \\ &=& (\tilde{D}_iK^i{}_j - \tilde{D}_jK)\tilde{e}^j + (\dot{K} + K^i{}_jK^j{}_i)du \\ R_i &=& E_v\lrcorner R^v{}_i + E_j\lrcorner R^j{}_i \\ &=& \tilde{R}_i + (-\tilde{E}_j\lrcorner\dot{\tilde{\omega}}^j{}_j + \tilde{D}_jF^j{}_i)du \\ &=& \tilde{R}_i + (\tilde{D}_jK^j{}_i - \tilde{D}_iK)du \end{array} \tag{18}$$

in accordance with the symmetry of the Ricci tensor. Imposing the vacuum equations results in

$$\begin{array}{rcl} \tilde{D}_iK^i{}_j - \tilde{D}_jK &=& 0 \\ \tilde{R}_i &=& 0 \\ \dot{K} + K^i{}_jK^j{}_i &=& 0 \end{array} \tag{19}$$

Switching back to the metric representation we find

$$\dot{\sigma}_{ab} = (\delta_{ij}\tilde{e}^i{}_a\tilde{e}^j{}_b)^\bullet = 2K_{ab} \tag{20}$$

which once again gives a split into evolution and constraint equations

$$\begin{aligned} \dot{\sigma}_{ab} = 2K_{ab} \qquad & D_a K^a{}_b - D_b K = 0 \\ \dot{K} + K^a{}_b K^b{}_a = 0 \qquad & \tilde{R}_{ab} = \frac{1}{2}\sigma_{ab}\tilde{R} = 0 \end{aligned} \tag{21}$$

(where D_a denotes the Levi-Civita connection associated with σ_{ab})
The Ricci constraint entails the flatness of the two-dimensional sections which in turn allows the explicit solution of the "momentum" constraint, via Fourier-transforms

$$K^a{}_b = D^a D_b \frac{1}{D^2} K, \tag{22}$$

where the action of the inverse of D^2 is given by the corresponding convolution with the Green-function of the two-dimensional Laplace operator.

3 Propagation of the constraints

In order to show that "time" evolution respects the constraint equations we will consider first the variation of the Ricci-scalar

$$\delta\tilde{R} = -\delta\sigma^{ab}\tilde{R}_{ab} + D_a D_b \delta\sigma^{ab} - D^2\delta\sigma \tag{23}$$

Taking the variation to be the "time"-derivative, i.e. $\delta\sigma_{ab} = \dot{\sigma}_{ab} = 2K_{ab}$ the above becomes

$$\begin{aligned} \delta\tilde{R} &= -2K^{ab}\tilde{R}_{ab} + 2D_a D_b K^{ab} - 2D^2 K \\ &= -K\tilde{R} + 2D_a(D_b K^{ba} - D^a K) \end{aligned} \tag{24}$$

which is zero if the constraints are fulfilled initially.

Let us now turn to the variation of the second constraint

$$\begin{aligned} \delta(D_a K^a{}_b - D_b K) &= \delta D_a K^a{}_b + D_a \delta K^a{}_b - D_b \delta K \\ &= \delta C^a{}_{ma} K^m{}_b - \delta C^m{}_{ba} K^a{}_m + D_a \delta K^a{}_b + D_b(K^{mn}K_{mn}) \end{aligned} \tag{25}$$

in order to evaluate the variation of $K^a{}_b$ we have to make use of the constraint to express it completely in terms of K, whose time-variation is given. Since

the two-dimensional slices are flat we may Fourier-transform the constraint, which turns the differential equation into an algebraic one. Its solution is given by

$$K^a{}_b = D^a D_b \frac{1}{D^2} K \tag{26}$$

where $1/D^2$ denotes the inverse of the Laplacian D^2. Using this expression let us first calculate the variation of $K^a{}_b$

$$\begin{aligned} \delta K^a{}_b &= -\delta\sigma^{ac} D_c D_b \frac{1}{D^2} K + \sigma^{ac} \delta D_c D_b \frac{1}{D^2} K \\ &\quad - D^a D_b \frac{1}{D^2} \delta D^2 \frac{1}{D^2} K + D^a D_b \frac{1}{D^2} \delta K \\ &= -2K^{ac} K_{cb} - \delta C^m{}_b{}^a D_m \frac{1}{D^2} K + D^a D_b \left(\frac{1}{D^2} (\delta\sigma^{cd} D_c D_d \frac{1}{D^2} K)\right) \\ &\quad + D^a D_b \frac{1}{D^2} (\sigma^{cd} \delta C^m{}_{dc} D_m \frac{1}{D^2} K) - D^a D_b \frac{1}{D^2} (K^{cd} K_{cd}) \end{aligned} \tag{27}$$

Taking into account that the difference tensor $\delta C^a{}_{bc}$ which determines the variation of the derivative operator D_a is completely determined by the variation of the metric $\delta\sigma_{ab}$

$$\begin{aligned} \delta C^a{}_{bc} &= \frac{1}{2} (D_b \delta\sigma^a{}_c + D_c \delta\sigma^a{}_b - D^a \delta\sigma_{bc}) \\ &= (D_b K^a{}_c + D_c K^a{}_b - D^a K_{bc}) = D_b K^a{}_c \end{aligned} \tag{28}$$

(where the last equality took the explicit form of $K^a{}_b$ in terms of K into account) the above becomes

$$\begin{aligned} &= -2K^{ac} K_{cb} - D_b K^{ma} D_m \frac{1}{D^2} K + 2 D^a D_b \frac{1}{D^2} (K^{cd} D_c D_d \frac{1}{D^2} K) \\ &\quad + D^a D_b \frac{1}{D^2} (D_c K^{mc} D_m \frac{1}{D^2} K) - D^a D_b \frac{1}{D^2} (K_{cd} K^{cd}) \\ &= -2K^{ac} K_{cb} - D^m K^a{}_b D_m \frac{1}{D^2} K + D^a D_b \frac{1}{D^2} (K^{cd} K_{cd}) \\ &\quad + D^a D_b \frac{1}{D^2} (D^m K D_m \frac{1}{D^2} K) \end{aligned} \tag{29}$$

Taking this result into account the variation of $D_a K^a{}_b - D_b K$ becomes

$$\begin{aligned}
\delta(D_a K^a{}_b - D_b K) &= \delta C^a{}_{ma} K^m{}_b - \delta C^m{}_{ba} K^a{}_m + D_a \delta K^a{}_b - D_b \delta K \\
&= D_m K K^m{}_b - D_b K^m{}_a K^a{}_m - 2D_a(K^{ac} K_{cb}) \\
&\quad - D_a(D^m K^a{}_b D_m \frac{1}{D^2} K) + 2D_b(K^{cd} K_{cd}) \\
&\quad + D_b(D_m K \frac{1}{D^2} D^m K) \\
&= 2D_m K K^m{}_b - 2D_b K^m{}_a K^a{}_m - 2D_a(K^{ac} K_{cd}) \\
&\quad + 2D_b(K^{cd} K_{cd}) \qquad (30) \\
&= 0
\end{aligned}$$

4 Hamiltonian dynamics

Since the Einstein-Hilbert action vanishes identically for pp-waves, which follows from $R_{ab} \propto p_a p_b$, the question about a Hamiltonian description does not seem to be a very sensible one. Nevertheless since the dynamical equations are non-trivial they may be taken as a starting point for the construction of symplectic structure as well as a Hamiltonian. In order to exhibit this point of view more explicitly let us consider electrodynamics first, i.e. try to construct a Hamiltonian description by starting from the Maxwell equations rather than the electromagnetic action.

The source-free Maxwell system

$$\begin{aligned}
\epsilon^{abc} D_b B_c - \dot{E}^a = 0 \qquad D_a B^a = 0 \\
\epsilon^{abc} D_b E_c + \dot{B}^a = 0 \qquad D_a E^a = 0
\end{aligned} \tag{31}$$

neatly splits into evolution and constraint equations. Introducing the vector potential A_a, which we will take as configuration variable

$$B^a = \epsilon^{abc} D_b A_c \tag{32}$$

solves the first constraint, at the price of being not unique. I.e.

$$A_a \longrightarrow A_a + D_a \Lambda \tag{33}$$

describes the same physical situation. In order to find the corresponding momentum we will take a little "quantum"–detour.

Let us assume that the (physical) wave-function $\Psi[A_a]$ is invariant[1] under gauge transformations, i.e.

$$\Psi[A_a + D_a\Lambda] = \Psi[A_a] \tag{34}$$

which, by the arbitrariness of Λ, is equivalent to

$$D_a \frac{\delta\Psi}{\delta A_a} = 0. \tag{35}$$

Identifying the derivative with respect to the configuration variable (up to a factor $1/i$) with the momentum(operator) suggests to identify the latter with E^a. Since we now have derived "position" and "momentum" variables we have constructed the symplectic form.

All that is left is to show that the evolution equations are Hamiltonian with respect to this symplectic form. From

$$\dot{E}^a = -\frac{\delta H}{\delta A_a} = \epsilon^{abc} D_b(\epsilon_{cmn} D^m A^n) \tag{36}$$

we find

$$\begin{aligned} \delta_A H &= -\int \delta A_a \epsilon^{abc} D_b(\epsilon_{cmn} D^m A^n)\omega_\delta \\ &= -\int \epsilon^{cba} D_b \delta A_a \epsilon_{cmn} D^m A^n \omega_\delta \\ &= -\delta \frac{1}{2} \int B_a B^a \omega_\delta \end{aligned} \tag{37}$$

(ω_δ denotes the volume form of $\mathbb{R}^3$) Whereas

$$\begin{aligned} \dot{B}^a &= \epsilon^{abc} D_b \dot{A}_c = -\epsilon^{abc} D_b E_c \\ 0 &= \epsilon^{abc} D_b(\dot{A}_c + E_c) \end{aligned} \tag{38}$$

entails

$$\dot{A}_a = \frac{\delta H}{\delta E^a} = -E_a + D_a\Lambda \tag{39}$$

where the last term arises from the kernel of $\epsilon^{abc} D_b$. Upon integration this yields

$$\delta_E H = -\delta \int (\frac{1}{2} E^a E_a + D_a E^a \Lambda)\omega_\delta \tag{40}$$

[1] This is actually a rather strong requirement, but it suffices for our purpose to identify the canonical momentum

Putting everything together we find for the Hamiltonian of the Maxwell system

$$H = -\frac{1}{2} \int (E_a E^a + B_a B^a + D_a E^a \Lambda) \tag{41}$$

which is the "correct" result, i.e. the one obtained from starting with the electromagnetic action.

Let us now apply this procedure to the pp-wave system

$$\begin{aligned} \dot{K} + K_{ab}K^{ab} = 0 \qquad & D_a K^a{}_b - D_b K = 0 \\ \dot{\sigma}_{ab} = 2K_{ab} \qquad & \tilde{R} = 0 \end{aligned} \tag{42}$$

The situation is very similar to the electromagnetic case. Again the system splits into dynamical and constraint equations. Therefore in the first step we will proceed by trying to identify the symplectic form. Let us begin by taking the 2-metric σ_{ab} as configuration variable (which is a step motivated from standard 3+1 ADM decomposition). In order to find the corresponding momentum we will require that the wave-function should be invariant under (infinitesimal) two-dimensional diffeomorphisms ξ^a, i.e.

$$\Psi[\sigma_{ab} + D_a \xi_b + D_b \xi_a] = \Psi[\sigma_{ab}] \tag{43}$$

This entails, due to the arbitrariness of ξ^a

$$D_a \frac{\delta \Psi}{\delta \sigma_{ab}} = 0. \tag{44}$$

Once again, since the derivative with respect to the configuration variable (up to a factor $1/i$) represents the momentum(operator) $\tilde{\pi}^{ab}$ this suggests to identify the latter with

$$\tilde{\pi}^{ab} = \omega_\sigma (K^{ab} - \sigma^{ab} K) \tag{45}$$

if we take the first constraint into account. (Note that momentum has to be tensor-valued 2-form, which can easily be seen from it being the derivative of the scalar Ψ with respect to the tensor σ_{ab}. In the following the two-form indices will be suppressed in favor of a tilde). Having identified position and momentum variables, which is equivalent to the identification of the symplectic structure it remains to show that the evolution relative to this symplectic structure is Hamiltonian. Taking into account that

$$\tilde{\pi} = -\omega_\sigma K \qquad \tilde{\pi} := \sigma_{ab} \tilde{\pi}^{ab} \tag{46}$$

the dynamical equations become

$$\begin{aligned} \dot{\tilde{\pi}} &= \omega_\sigma^{-1}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tilde{\pi}^2) \\ \dot{\sigma}_{ab} &= 2\omega_\sigma^{-1}(\tilde{\pi}_{ab} - \sigma_{ab}\tilde{\pi}) \end{aligned} \tag{47}$$

in terms of the canonical variables. (Here the expression ω_σ^{-1} denotes the inverse volume form of the 2-slice, i.e. locally $\omega_\sigma^{-1} = 1/\sqrt{\sigma}\partial_1 \wedge \partial_2$) Integration of the second equation of motion gives

$$\dot{\sigma}_{ab} = \frac{\delta H}{\delta \tilde{\pi}^{ab}} \qquad \delta_\pi H = \int 2\omega_\sigma^{-1}\delta\tilde{\pi}^{ab}(\tilde{\pi}_{ab} - \sigma_{ab}\tilde{\pi}) = \delta_\pi \int \omega_\sigma^{-1}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tilde{\pi}^2). \tag{48}$$

Let us now derive the variation of π. Taking into account that $\tilde{\pi} = \sigma_{ab}\tilde{\pi}^{ab}$ we have

$$\begin{aligned} \dot{\tilde{\pi}} &= \dot{\sigma}_{ab}\tilde{\pi}^{ab} + \sigma_{ab}\dot{\tilde{\pi}}^{ab} = 2\omega_\sigma^{-1}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tilde{\pi}^2) - \sigma_{ab}\frac{\delta H}{\delta \sigma_{ab}} \\ &= 2\omega_\sigma^{-1}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tilde{\pi}^2) - \sigma_{ab}(2\omega_\sigma^{-1}(\tilde{\pi}^{ac}\tilde{\pi}_c{}^b - \tilde{\pi}^{ab}\tilde{\pi}) - \frac{1}{2}\omega_\sigma^{-1}\sigma^{ab}(\tilde{\pi}^{cd}\tilde{\pi}_{cd} - \tilde{\pi}^2)) \\ &= \omega_\sigma^{-1}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tilde{\pi}^2) \end{aligned} \tag{49}$$

where the expression for H has been taken from the previous. Since the result coincides with the first equation of motion we may take

$$H[\sigma, \tilde{\pi}] = \int \omega_\sigma^{-1}(\tilde{\pi}^{ab}\tilde{\pi}^{cd}\sigma_{ac}\sigma_{bd} - (\tilde{\pi}^{cd}\sigma_{cd})^2) \tag{50}$$

to be the Hamiltonian of the our system.

Conclusion

We have shown that it is possible to formulate the dynamics of the pp-wave system similar to the Gaussian evolution of the standard timelike situation. At first sight the vanishing action, i.e. its topological nature, seems to hamper a Hamiltonian formulation. Nevertheless upon comparison with the electromagnetic system we succeed in identifying both symplectic structure as well as the dynamical Hamilton function. We believe that this opens the road to the quantization of the model in terms of a midi-superspace formulation. Work in this direction is currently in progress.

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